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Finite element formulation for dynamics of delaminated composite beams with piezoelectric actuators

Victor Y. Perel [a], Anthony N. Palazotto [b,*]

[a] *AFRL/MLPS, Bldg. 651, 3005 P. Street, Wright-Patterson Air Force Base, Ohio 45433, USA*
[b] *Air Force Institute of Technology, AFIT/ENY, Wright-Patterson Air Force Base, Ohio 45433-7765, USA*

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Abstract

As a part of an effort to develop a model-supported method for detection of delaminations in composite beams with the use of time responses to external excitations, a finite element formulation for dynamics of a composite beam with delamination and attached piezoelectric actuators is developed. In this formulation account is taken of transverse shear deformation and nonlinear through-thickness variation of the longitudinal displacement. Parameters that characterize the delamination are incorporated into the formulation that makes the finite element model convenient for use in conjunction with damage identification (not discussed in the present paper). Computational predictions of frequencies show good agreement with experimental results.
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1. Introduction

Mechanical properties of composite materials can degrade severely in the presence of damage. One of common types of damage modes in laminated composites is interlaminar cracking or delamination. Delaminations may develop as a result of manufacturing defects or impact of foreign objects. Delaminations are known to cause a change in vibration characteristics of composite structures that can be used to detect a presence of the delamination and estimate its size and location. This can be done by a finite element model updating to obtain a correct set of physical parameters, characterizing the delamination, that minimize some measure of discrepancy between the vibration data measured experimentally and obtained from the finite element model of the delaminated structure.

Some aspects of theory of the model-aided damage detection based on changes of dynamic characteristics of structures and examples of implementation of the method (for types of damage not related to delamination) are presented by Natke and Cempel (1997). A review of literature, published before 1997,

* Corresponding author. Tel.: +1-937-255-3636; fax: +1-937-656-7621.
E-mail addresses: victor.perel@wpafb.af.mil (V.Y. Perel), anthony.palazotto@afit.edu (A.N. Palazotto).

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devoted to the subject, which includes detection of delaminations, was published by Doebling et al. (1996). More recently, the papers regarding the model-based delamination detection were published by Hanagud with co-authors (for example, Lestari and Hanagud, 1999) and by Chattopadhyay with co-authors (for example, Chattopadhyay et al., 2000). So, the model-based structural health monitoring with the use of dynamic response is an active area of research, which requires further development.

The model-aided method of detection of delaminations requires a highly accurate finite element model of the delaminated structure that contains parameters characterizing delaminations. In order to be accurate, such a model of a delaminated beam must have a capability to take account of transverse shear deformation, nonlinear through-the-thickness variation of longitudinal displacements, through-the-thickness continuity of displacements in sublaminates that do not contain delaminations, discontinuity of displacements at the surfaces of the delamination crack, must satisfy stress boundary conditions on the upper and lower surfaces of the beam, a condition of vanishing of the transverse stresses on the surfaces of the delamination crack and a condition of continuity of the transverse stresses at the interfaces between the plies with different material properties and fiber orientations.

Finite element models for accurate analysis of delaminated plates and beams were created, for example, by Barbero and Reddy (1991) and Seeley and Chattopadhyay (1999). However, as we understand, in these models, in order to change a location and size of the delamination crack, a new finite element mesh has to be constructed, while in our model, presented in this paper, in order to change a location and size of the crack, one needs only to change values of certain parameters (coordinates of crack tips and distance of the crack from the middle surface) on which components of stiffness and mass matrices depend. Therefore, in order to use the other authors' (referred to) models in model-aided experimental damage identification procedures, one has to construct a large number of finite element meshes in order to choose among them the one, which reproduces experimental data as closely as possible. The same is true for modelling delaminations with commercially available finite element codes. In addition, the Reddy's theory for plates with delaminations is based on a discrete-layer approach, and, therefore, is equivalent to a three-dimensional (3-D) finite element approach, requiring a large number of degrees of freedom in order to model a laminated plate or a beam. Therefore, in our opinion, the use of the other authors' models, as well as the commercial finite element codes, is not convenient for the model-based health monitoring of structures.

The model of a delaminated beam, presented in this paper, is developed for the use in damage identification procedure based on comparing the delaminated beam's experimentally measured and computed time responses to excitation from piezoelectric actuators attached to the beam, [1] similar to the method proposed by Banks et al. (1996) for detection of holes in beams. Therefore, in addition to the finite element for beam's segment with delamination, we develop a finite element for a beam's segment with a piezoelectric actuator on its upper surface. It is assumed, for simplicity, that the segment of the beam covered with the piezoelectric patch (actuator) does not contain the delamination.

In the present paper we construct a finite element formulation for a beam with a through-width delamination, which satisfies the above-mentioned requirements, in such a way that components of the stiffness and mass matrices depend on three parameters that characterize the location and length of the delamination crack, and the values of these parameters do not have to be known in order to construct a finite element mesh. These parameters are coordinates of the tips of the delamination crack, which is assumed to be parallel to the surfaces of the beam. In the finite element analysis of the beam with known location and length of the delamination crack (direct problem), these parameters are known; but in the

[1] In this paper, the authors' goal is to describe the formulation and solution of the problem of analysis of the delaminated beam with the known location of the delamination (direct problem) and its experimental verification. A description of application of this formulation to the solution of the problem of the model-based detection of delamination (inverse problem) will be published elsewhere.

inverse problem of damage identification these parameters are treated as unknowns which are to be computed by minimizing a certain function that characterizes the discrepancy between the computed and measured time responses of delaminated beams.

For an initial formulation, we consider only the direct problem of analysis of the beam with the known location and extent of the delamination crack. Thus, methods of damage identification with the use of the developed finite element formulation of the delaminated beam are not presented in this paper.

2. Three-dimensional formulation

The 3-D formulation of the problem of dynamics of the composite beam with the delamination crack and with piezoelectric layers (used as actuators) include the following equations.

Strain–displacement equations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \varepsilon_{yy} = \frac{\partial u}{\partial y}, \tag{1a}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}; \tag{1b}$$

Equations of motion:

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} = \rho\ddot{u}, \qquad \sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} = \rho\ddot{v}, \qquad \sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} - \rho g = \rho\ddot{w}. \tag{2}$$

We will consider a piezoelectric material, used in the actuator, with orthorhombic mm2 symmetry, such as polyvinylidene (PVDF) or lead zirconate-titanate (PZT). In the manufacturing process, the planes of elastic symmetry can be made the same as the planes of piezoelectric symmetry. In this case, the constitutive relations in the principle material coordinate system have the form (Mitchell and Reddy, 1994; Varadan et al., 1989; Nix and Ward, 1986; Dunn and Taya, 1993):

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 & 0 & 0 & e_{32} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 & 0 & 0 & e_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & e_{24} & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 & e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \xi_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 & 0 & \xi_{22} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 & 0 & 0 & \xi_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \end{Bmatrix}, \tag{3}$$

where $\mathscr{E}_k$ are components of the electric field intensity applied to the piezoelectric element; D_i are components of electric displacement, and ξ_{ij} and e_{ij} are constants that characterize electromechanical properties of piezoelectric materials (Appendix A). For an orthotropic composite material of the beam that does not have piezoelectric properties, $e_{ij} = 0$ and $\xi_{ij} = 0$.

An approximation that the electric field intensity in the piezoelectric actuator under low-frequency applied voltage is derivable only from a scalar electric potential (Tiersten, 1969) leads to

$$\mathscr{E}_x = -\frac{\partial \varphi}{\partial x}, \qquad \mathscr{E}_y = -\frac{\partial \varphi}{\partial y}, \qquad \mathscr{E}_z = -\frac{\partial \varphi}{\partial z}. \tag{4}$$

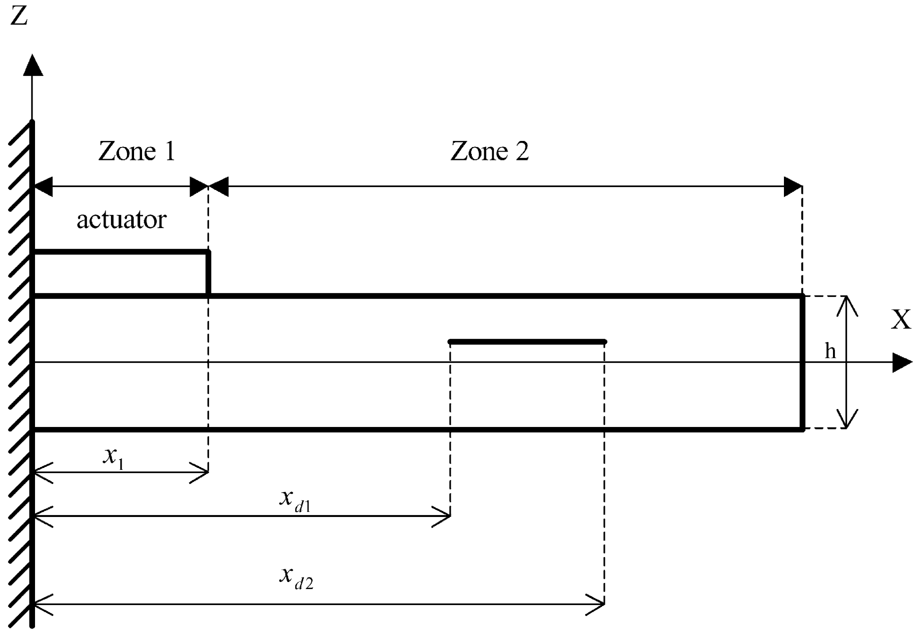

Fig. 1. Cantilever beam with delamination and actuator.

We consider a beam with no externally applied loads on the upper and lower surfaces, [2] therefore stress boundary conditions on the upper and lower surfaces are

$$\sigma_{xz} = 0 \quad \text{at } z = \pm\frac{h}{2}, \tag{5a}$$

$$\sigma_{zz} = 0 \quad \text{at } z = \pm\frac{h}{2}. \tag{5b}$$

Stress boundary conditions at the location of the delamination:

$$\sigma_{xz} = 0 \quad \text{at } z = z_{\mathrm{d}}, \quad x_{\mathrm{d1}} \leqslant x \leqslant x_{\mathrm{d2}}, \tag{6}$$

$$\sigma_{zz} = 0 \quad \text{at } z = z_{\mathrm{d}}, \quad x_{\mathrm{d1}} \leqslant x \leqslant x_{\mathrm{d2}}, \tag{7}$$

where z_{d} is a z-coordinate of the delamination crack, which is parallel to the surfaces of the beam, x_{d1} and x_{d2} are x-coordinates of the tips of the delamination crack (Fig. 1). For a beam clamped at the edge $x = 0$, the boundary conditions at the contour of the beam are:

boundary conditions at the clamped edge are:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, \tag{8}$$

boundary conditions for the stress-free edges can be stated as

[2] Excitation of vibrations of the beam is done with the use of a piezoelectric actuator, attached (glued) to the beam, therefore the actuator is considered to be a part of the beam, and the force of interaction between the actuator and the beam is not a beam's externally applied load.

$$\sigma_{xx} = 0, \quad \sigma_{xy} = 0, \quad \sigma_{xz} = 0 \quad \text{at } x = L; \qquad \sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad \sigma_{zy} = 0 \quad \text{at } y = \pm\frac{b}{2}. \tag{9}$$

Besides, we have to take into account that displacements can be discontinuous at the location of the delamination, i.e. at $z = z_d$, $x_{d1} \leqslant x \leqslant x_{d2}$.

3. Development of the one-dimensional beam theory

We are considering a composite cantilever beam with one delamination crack, parallel to the surfaces of the beam, and one piezoelectric patch, used as an actuator, attached near the clamped edge of the beam. Two separate types of finite elements will be developed:

1. for the zone with the piezoelectric actuator attached to the upper surface and without delaminations (zone 1 in Fig. 1);
2. for the zone without actuators and with the delamination crack (zone 2 in Fig. 1).

Zone 2 includes a region with the delamination crack ($x_{d1} \leqslant x \leqslant x_{d2}$) and two regions without delamination cracks ($x_1 \leqslant x < x_{d1}$, $x_{d2} < x \leqslant L$). In order to analyze a whole beam with the actuator and the delamination crack, both types of elements will be included into the finite element mesh.

3.1. Simplifying assumptions of the beam theory

The following simplifying assumptions are adopted in order to reduce the 3-D formulation of the problem to the 1-D beam-type formulation.

Assumption 1. If a beam is narrow in both the y-direction and the z-direction, and not loaded in these directions, then stresses σ_{yy} and σ_{zz}, can be set equal to zero:

$$\sigma_{yy} = 0, \qquad \sigma_{zz} = 0. \tag{10}$$

Under this assumption, the requirement that $\sigma_{zz} = 0$ at upper and lower surfaces of the beam (Eq. (5b)) and at the location of the delamination (Eq. (7)) is satisfied.

Assumption 2. In our problem, the external electric field will be applied to the piezoelectric actuators only in the z-direction (perpendicular to the planes of piezoelectric films). Therefore,

$$\mathscr{E}_x = \mathscr{E}_y = 0. \tag{11}$$

Assumption 3. If at each point of the actuator and the beam there is a plane of elastic symmetry parallel to the x–z plane (for the beam this can occur in case of [0°/90°] ply lay-up), the transverse load intensity, q_z, does not vary in the y-direction, and there is no load in the y-direction, $q_y = 0$, then (Lekhnitskii, 1963)

$$v = 0, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial w}{\partial y} = 0. \tag{12a}$$

If, in addition, a material of the piezoelectric actuator has an orthorhombic mm2 symmetry (PVDF or lead PZT) and its planes of piezoelectric symmetry are aligned with the planes of elastic symmetry of itself and of the composite material of the beam, then the piezoelectric constant $\bar{\varrho}_{36}$ is equal to zero,

$$\bar{\varrho}_{36} = 0, \tag{12b}$$

due to transformation equation $\bar{\varrho}_{36} = (\varrho_{31} - \varrho_{32})\cos\theta\sin\theta$ (Appendix A, Eq. (A.17)), where, in view of our assumptions, $\theta = 0°$ or $\theta = 90°$. In this case, the two other stresses associated with the y-direction are equal to zero:

$$\sigma_{xy} = \sigma_{yz} = 0. \tag{13}$$

Indeed,

$$\text{if } v = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad \text{then } \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0,$$

and, therefore, according to Eqs. (A.16) and (A.17) in Appendix A,

$$\sigma_{xy} = \overline{Q}_{66} 2 \underbrace{\varepsilon_{xy}}_{0} + \underbrace{\bar{\varrho}_{36}}_{0} \mathscr{E}_z = 0, \qquad \sigma_{yz} = \overline{Q}_{44} 2 \underbrace{\varepsilon_{yz}}_{0} + \underbrace{\overline{Q}_{45}}_{0} 2\varepsilon_{xz} + \bar{\varrho}_{14} \underbrace{\mathscr{E}_x}_{0} + \bar{\varrho}_{24} \underbrace{\mathscr{E}_y}_{0} = 0.$$

In the last equation, the elastic coefficient $\overline{Q}_{45}$ is set equal to zero for the composite beam with [0°/90°] ply lay-up in view of the transformation equation $\overline{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta$ (Appendix A, Eq. (A.17)).

In view of Eqs. (10)–(13), the constitutive equations for a material of the piezoelectric actuator attached to a narrow beam, in a problem coordinate system, take the form (Appendix A, Eq. (A.36)):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \frac{1}{\overline{S}_{11}} & 0 & \frac{\bar{d}_{31}}{\overline{S}_{11}} \\ 0 & \frac{1}{\overline{S}_{55}} & 0 \\ \frac{\bar{d}_{31}}{\overline{S}_{11}} & 0 & \left(-\bar{\zeta}_{33} + \frac{\bar{d}_{31}^2}{\overline{S}_{11}}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \frac{\partial\varphi}{\partial z} \end{Bmatrix}, \tag{14}$$

where φ, is the scalar electric potential, defined by formula (4); the compliance coefficients in the problem coordinate system, $\overline{S}_{11}$ and $\overline{S}_{55}$, are expressed in terms of the engineering constants by the formulas (Appendix A, Eqs. (A.32) and (A.33))

$$\overline{S}_{11} = \frac{1}{E_1}c^4 + \frac{1}{E_2}s^4 + \left(\frac{1}{G_{12}} - 2\frac{\nu_{12}}{E_1}\right)s^2c^2, \qquad \overline{S}_{55} = \frac{1}{G_{23}}s^2 + \frac{1}{G_{13}}c^2 \tag{15}$$

($c = \cos\theta$, $s = \sin\theta$, θ is an angle of fiber orientation); the constants $\bar{d}_{31}$ and $\bar{d}_{35}$, which characterize the piezoelectric properties in the problem coordinate system are expressed in terms of the constants d_{ij} in the material coordinate system by the formulas (Appendix A, Eqs. (A.34) and (A.35))

$$\bar{d}_{31} = d_{31}c^2 + d_{32}s^2 - d_{36}sc, \qquad \bar{d}_{35} = -d_{34}s + d_{35}c, \qquad \bar{\zeta}_{33} = \zeta_{33}. \tag{16}$$

For a composite material of the beam, where the piezoelectric constants are equal to zero, the constitutive equation (14) have the form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\overline{S}_{11}} & 0 \\ 0 & \frac{1}{\overline{S}_{55}} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \end{Bmatrix}. \tag{17}$$

The strain–displacement relations (Eq. (1a) and (1b)) for this simplified 1-D problem, due to Eq. (12a), take the form

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad \varepsilon_{xy} = 0, \qquad \varepsilon_{yy} = 0, \qquad \varepsilon_{yz} = 0, \qquad \varepsilon_{zz} = 0. \tag{18}$$

The delamination occupies a region $x_{d1} \leqslant x \leqslant x_{d2}$, and is assumed to be above the middle surface of the beam (where $z = 0$), at a location with z-coordinate $z = z_d > 0$. It is also assumed that the delamination extends through all the width of the beam, $-b/2 \leqslant y \leqslant b/2$.

Assumption 4. A very common assumption that the beam's height does not change during deformation is adopted here:

$$\varepsilon_{zz} = 0. \tag{19}$$

This assumption in conjunction with the strain–displacement relation $\varepsilon_{zz} = \partial w/\partial z$ and the requirement of taking account of discontinuity of a displacement w at the location of delamination ($z = z_{\rm d}$, $x_{\rm d1} \leqslant x \leqslant x_{\rm d2}$), leads to the following assumed displacement w in the region with the delamination ($x_{\rm d1} \leqslant x \leqslant x_{\rm d2}$), assuming that the delamination crack is above the middle surface of the plate ($z_{\rm d} > 0$):

$$w(x,y,z,t) = [1 - H(z)]W_0(x,y,t) + H(z)W_1(x,y,t) \quad (x_{\rm d1} \leqslant x \leqslant x_{\rm d2}), \tag{20}$$

where $W_0(x,y,t)$ is a transverse displacement (in z-direction) of the middle surface of the beam (where $z = 0$), $W_1(x,y,t)$ is the transverse displacement of the upper surface of the delamination crack, and $H(z)$ is a Heaviside function that can be defined as

$$H(z) \equiv H(z,z_{\rm d}) \equiv \lim_{\beta\to 0} \frac{1}{\pi}\arctan\frac{z - z_{\rm d}}{\beta} + \frac{1}{2} = \frac{1}{2}\mathrm{signum}(z - z_{\rm d}) + \frac{1}{2} = \begin{cases} 0 & \text{for } z < z_{\rm d}, \\ 0.5 & \text{for } z = z_{\rm d}, \\ 1 & \text{for } z > z_{\rm d}, \end{cases} \tag{21}$$

where the function signum(z) is defined as follows

$$\mathrm{signum}(z) = \begin{cases} 0 & \text{for } z = 0, \\ \frac{z}{|z|} & \text{for all other } z. \end{cases} \tag{22}$$

In the region without delamination ($0 \leqslant x < x_{\rm d1}$, $x_{\rm d1} < x \leqslant L$), the assumed displacement has the form

$$w(x,y,z,t) = W_0(x,y,t) \quad (0 \leqslant x < x_{\rm d1},\ x_{\rm d1} < x \leqslant L). \tag{23}$$

Assumption 5. From constitutive equations (14) we have

$$\sigma_{xz} = \frac{1}{\bar{S}_{55}} 2\varepsilon_{xz}. \tag{24}$$

Therefore, in order to satisfy the stress boundary conditions (6), i.e. vanishing of the transverse stress σ_{xz} at the outer surfaces of the plate and the surfaces of the delamination crack, the assumed transverse shear strain ε_{xz} must also vanish at the same surfaces. Besides, in order to represent the through-the-thickness variation of the transverse stress σ_{xz} realistically, we will assume that this stress varies quadratically in the thickness direction of the plate. Then, according to the constitutive equation (24), the same through-the-thickness variation of the transverse strain ε_{xz} must be assumed. This leads to the following simplifying assumptions about variation of the transverse strains in the thickness direction:

For the region of the zone 2 with the delamination ($x_{\rm d1} \leqslant x \leqslant x_{\rm d2}$) and without attached or embedded piezoelectric actuators (Fig. 1), it is assumed that

$$2\varepsilon_{xz}(x,y,z,t) = \varphi_x^{(1)}(x,y,t)\left(1 + \frac{2}{h}z\right)\left(1 - \frac{z}{z_{\rm d}}\right)[1 - H(z)] + \varphi_x^{(2)}(x,y,t)\left(-1 + \frac{2z}{h}\right)\left(1 - \frac{z}{z_{\rm d}}\right)H(z) \quad (x_{\rm d1} \leqslant x \leqslant x_{\rm d2}), \tag{25}$$

where $\varphi_x^{(1)}(x,y,t)$ is an unknown function that characterizes the strain ε_{xz} under the delamination crack ($-h/2 \leqslant z < z_{\rm d}$, $x_{\rm d1} \leqslant x \leqslant x_{\rm d2}$) and $\varphi_x^{(2)}(x,y,t)$ is the unknown functions that characterizes the strain ε_{xz} above the delamination crack ($z_{\rm d} < z \leqslant h/2$, $x_{\rm d1} \leqslant x \leqslant x_{\rm d2}$).

For the region of the zone 2 without delaminations ($x_1 \leqslant x < x_{\rm d1}$, $x_{\rm d1} < x \leqslant L$) and without attached or embedded piezoelectric actuators (Fig. 1), it is assumed that

$$2\varepsilon_{xz}(x,y,z,t) = \varphi_x(x,y,t)\left(1-\frac{2}{h}z\right)\left(1+\frac{2}{h}z\right) \quad (x_1 \leqslant x < x_{\mathrm{d}1},\ x_{\mathrm{d}2} \leqslant x \leqslant L), \tag{26}$$

where $\varphi_x(x,y,t)$ is an unknown function that characterizes the strain ε_{xz} in the region without delaminations.

For the zone 1 of the beam, i.e. for the zone without delaminations and with the piezoelectric actuator attached to the upper surface it is assumed that

$$2\varepsilon_{xz} = \psi_x(x,t)\left(1-\frac{2}{h+2t}z\right)\left(1+\frac{2}{h}z\right) \quad 0 \leqslant x < x_1, \tag{27}$$

where t is the thickness of the piezoelectric patch, and $\psi_x(x,t)$ is an unknown function that characterizes the strain ε_{xz} in the region without delaminations and with the piezoelectric actuator attached to the upper surface of the beam. This assumed strain vanishes at the lower surface of the beam $z = -h/2$ and at the upper surface of the actuator, attached to the upper surface of the beam, i.e. at $z = (h/2) + t$ (Fig. 1).

3.2. Finite element formulation for the zone 1 of the beam (with the piezoelectric actuator attached to the upper surface and without delaminations)

Let u_0 be a longitudinal displacement at the axis of the beam (at $z = 0$), i.e. $u_0(x,t) \equiv u|_{z=0}$. In order to express the longitudinal displacement $u(x,z,t)$ in terms of the unknown functions $u_0(x,t)$, $\psi_x(x,t)$ and $W_0(x,t)$, we will integrate the strain–displacement relation $2\varepsilon_{xz} = \partial u/\partial z + \partial w/\partial x$ with the result

$$\begin{aligned} u(x,z,t) - U_0(x,t) &= \int_0^z \frac{\partial u}{\partial z}\,\mathrm{d}z = \int_0^z \left(2\varepsilon_{xz} - \frac{\partial w}{\partial x}\right)\mathrm{d}z \\ &= \int_0^z \left[\psi_x(x,t)\left(1-\frac{2}{h+2t}z\right)\left(1+\frac{2}{h}z\right) - \frac{\partial W_0(x,t)}{\partial x}\right]\mathrm{d}z \\ &= \psi_x(x,t)\frac{(-4z^2+6tz+3h^2+6ht)z}{3(h+2t)h} - \frac{\partial W_0(x,t)}{\partial x}z, \end{aligned} \tag{28}$$

or

$$u(x,z,t) = \begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 1 \\ 0 & 0 & \frac{2t}{(h+2t)h} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h} \end{bmatrix} \begin{Bmatrix} U_0 \\ W_0 \\ \psi_x \end{Bmatrix}. \tag{29}$$

With the use of Eq. (28) and the strain–displacement relation

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},$$

we can find the strain ε_{xx} in terms of the unknown functions:

$$\varepsilon_{xx} = \begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{2t}{(h+2t)h}\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h}\frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} U_0 \\ W_0 \\ \psi_x \end{Bmatrix}. \tag{30}$$

In this model, the virtual work principle for a finite element of a beam with a piezoelectric patch has the form:

$$\iiint_{(V_{\text{beam}})} \left(\sigma_{xx}^{(\text{b})}\delta\varepsilon_{xx} + \sigma_{xz}^{(\text{b})}2\delta\varepsilon_{xz}\right)\text{d}V + \iiint_{(V_{\text{patch}})} \left(\sigma_{xx}^{(\text{p})}\delta\varepsilon_{xx} + \sigma_{xz}^{(\text{p})}2\delta\varepsilon_{xz}\right)\text{d}V + \iiint_{(V)} \rho\ddot{u}\,\delta u\,\text{d}V + \iiint_{(V)} \rho(g+\ddot{w})\,\delta w\,\text{d}V = 0, \tag{31a}$$

where $(V) = (V_{\text{beam}}) + (V_{\text{patch}})$ is a volume of the whole finite element (beam and piezoelectric patch). Eq. (31a) can be written in the form

$$b\int_0^l \int_{-h/2}^{h/2} \left(\sigma_{xx}^{(\text{b})}\delta\varepsilon_{xx} + \sigma_{xz}^{(\text{b})}2\delta\varepsilon_{xz}\right)\text{d}z\,\text{d}x + b\int_0^l \int_{h/2}^{t} \left(\sigma_{xx}^{(\text{p})}\delta\varepsilon_{xx} + \sigma_{xz}^{(\text{p})}2\delta\varepsilon_{xz}\right)\text{d}z\,\text{d}x + b\int_0^l \int_{-h/2}^{t} \rho\ddot{u}\delta u\,\text{d}z\,\text{d}x + b\int_0^l \int_{-h/2}^{t} \rho(g+\ddot{w})\delta w\,\text{d}z\,\text{d}x = 0. \tag{31b}$$

According to the constitutive equation (A.36),

$$\sigma_{xz}^{(\text{b})} = \frac{1}{\overline{S}_{55}^{(\text{b})}}2\varepsilon_{xz}, \quad \sigma_{xx}^{(\text{b})} = \frac{1}{\overline{S}_{11}^{(\text{b})}}\varepsilon_{xx}, \quad \sigma_{xz}^{(\text{p})} = \frac{1}{\overline{S}_{55}^{(\text{p})}}2\varepsilon_{xz}, \quad \sigma_{xx}^{(\text{p})} = \frac{1}{\overline{S}_{11}^{(\text{p})}}\varepsilon_{xx} + \frac{\bar{d}_{31}}{\overline{S}_{11}}\frac{\partial\varphi}{\partial z} \approx \frac{1}{\overline{S}_{11}^{(\text{p})}}\varepsilon_{xx} - \frac{\bar{d}_{31}}{\overline{S}_{11}}\frac{V}{t}, \tag{32}$$

where the superscript (b) where stands for a composite material of the beam, and the superscript (p) stands for a material of the piezoelectric patch.

Substitution of constitutive equation (32) into the virtual work principle, Eq. (31b), yields

$$b\int_0^l \int_{-h/2}^{h/2} \left[\frac{1}{\overline{S}_{11}^{(\text{b})}}\varepsilon_{xx}\delta\varepsilon_{xx} + \frac{1}{\overline{S}_{55}^{(\text{b})}}2\varepsilon_{xz}2\delta\varepsilon_{xz}\right]\text{d}z\,\text{d}x + b\int_0^l \int_{h/2}^{t} \left[\left(\frac{1}{\overline{S}_{11}^{(\text{p})}}\varepsilon_{xx} - \frac{\bar{d}_{31}}{\overline{S}_{11}^{(\text{p})}}\frac{V}{t}\right)\delta\varepsilon_{xx} + \frac{1}{\overline{S}_{55}^{(\text{p})}}2\varepsilon_{xz}2\delta\varepsilon_{xz}\right]\text{d}z\,\text{d}x$$
$$+ b\int_0^l \int_{-h/2}^{h/2} \rho^{(\text{b})}\ddot{u}\delta u\,\text{d}z\,\text{d}x + b\int_0^l \int_{h/2}^{t} \rho^{(\text{p})}\ddot{u}\delta u\,\text{d}z\,\text{d}x + b\int_0^l \int_{-h/2}^{h/2} \rho^{(\text{b})}(g+\ddot{w})\delta w\,\text{d}z\,\text{d}x$$
$$+ b\int_0^l \int_{h/2}^{t} \rho^{(\text{p})}(g+\ddot{w})\delta w\,\text{d}z\,\text{d}x = 0. \tag{33}$$

Substitution of expressions for the displacements and strains in terms of the unknown functions, Eqs. (23), (27), (29) and (30), into the virtual work principle, Eq. (33), gives the virtual work principle in terms of the unknown functions in which the maximum orders of the derivatives of the unknown functions $U_0(x,t)$, $W_0(x,t)$ and $\psi_x(x,t)$ are

Function	Maximum order of derivatives
U_0	1
W_0	2
ψ_x	1

In order to perform a finite element formulation, we will represent the unknown functions $U_0(x,t)$, $W_0(x,t)$ and $\psi_x(x,t)$ by piecewise interpolation polynomials. If the virtual work principle contains spatial derivatives of a field variable with a highest order being a number m, then an interpolation polynomial must be chosen to satisfy the following requirements (Cook et al., 1989): (1) it must be a complete polynomial of degree m or higher; (2) across boundaries between elements, the interpolation polynomial and its derivatives through order $m-1$ (or higher) must be continuous. In the problem under consideration, the interpolation polynomial will be chosen to be of the lowest allowable degree and the order of their derivatives' continuity. Therefore, we choose the first degree Lagrange polynomials to interpolate the unknown functions U_0 and ψ_x

$$U_0 = \underset{(1\times2)}{\lfloor M \rfloor}\underset{(2\times1)}{\{\overline{U_0}\}}, \qquad \psi_x = \underset{(1\times2)}{\lfloor M \rfloor}\underset{(2\times1)}{\left\{\bar{\psi}_x\right\}}, \tag{34}$$

where

$$\lfloor M \rfloor \equiv \lfloor M_1 \quad M_2 \rfloor, \quad M_1 = 1 - \frac{\bar{x}}{l}, \;\; M_2 = \frac{\bar{x}}{l}, \tag{35}$$

$$\{\overline{U_0}\} \equiv \left\{ \begin{array}{c} U_0(0) \\ U_0(l) \end{array} \right\}, \qquad \left\{\bar{\psi}_x\right\} \equiv \left\{ \begin{array}{c} \psi_x(0) \\ \psi_x(l) \end{array} \right\}. \tag{36}$$

Here $\bar{x}$ is an x-coordinate in a local (element) coordinate system, whose origin coincides with the left node of the finite element, and l is a length of a finite element. Following the same rules, we choose the Hermit polynomials of the third degree to interpolate W_0:

$$W_0 = \underset{(1\times4)}{\lfloor N \rfloor}\underset{(4\times1)}{\{\overline{W}_0\}}, \tag{37}$$

where

$$\underset{(1\times4)}{\lfloor N \rfloor} \equiv \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor, \tag{38}$$

$$N_1 = 1 - \frac{3\bar{x}^2}{l^2} + \frac{2\bar{x}^3}{l^3}, \qquad N_2 = \bar{x} - \frac{2\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2}, \qquad N_3 = \frac{3\bar{x}^2}{l^2} - \frac{2\bar{x}^3}{l^3}, \qquad N_4 = -\frac{\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2}, \tag{39}$$

$$\underset{(4\times1)}{\{\overline{W}_0\}} = \left\{ \begin{array}{c} W_0(0) \\ \frac{\partial W_0}{\partial x}(0) \\ W_0(l) \\ \frac{\partial W_0}{\partial x}(l) \end{array} \right\}. \tag{40}$$

By introducing a column-matrix of nodal parameters $\{\tilde{\theta}\}_{(8\times1)}$, the components of which are defined as

$$\tilde{\theta}_1 = U_0(0), \quad \tilde{\theta}_2 = W_0(0), \quad \tilde{\theta}_3 = \frac{\partial W_0}{\partial x}(0), \quad \tilde{\theta}_4 = \psi_x(0), \quad \tilde{\theta}_5 = U_0(l), \quad \tilde{\theta}_6 = W_0(l),$$
$$\tilde{\theta}_7 = \frac{\partial W_0}{\partial x}(l), \quad \tilde{\theta}_8 = \psi_x(l) \tag{41}$$

and by substitution of polynomial approximations of the unknown functions (Eqs. (34) and (37)) into the virtual work principle for a finite element, written in terms of the unknown functions, one can obtain the virtual work principle for a finite element in terms of the nodal parameters:

$$\underset{(1\times8)}{\left\{\delta\tilde{\theta}\right\}^{\mathrm{T}}}\left(\underset{(8\times8)}{\left[\tilde{m}\right]}\underset{(8\times1)}{\left\{\ddot{\tilde{\theta}}\right\}} + \underset{(8\times8)}{\left[\tilde{k}\right]}\underset{(8\times1)}{\left\{\tilde{\theta}\right\}} - \underset{(8\times1)}{\{q\}}V(\tau)\right) = 0, \tag{42}$$

where expressions for matrices $[\tilde{m}]$, $[\tilde{k}]$ and vector $\{q\}$ are presented in Appendix B. Equations of motion of a finite element for the zone 1 of the beam, i.e. for the zone without delaminations and with the piezoelectric actuator attached to the upper surface (Fig. 1), follow from Eq. (42):

$$\underset{(8\times8)}{[\tilde{m}]}\underset{(8\times1)}{\left\{\ddot{\tilde{\theta}}\right\}}+\underset{(8\times8)}{[\tilde{k}]}\underset{(8\times1)}{\left\{\tilde{\theta}\right\}}=\underset{(8\times1)}{\{\tilde{q}\}}V(\tau). \tag{43}$$

3.3. Finite element formulation for the zone 2 of the beam (without actuators and with the delamination crack)

In the zone 2 of the beam (Fig. 1) there are no attached piezoelectric actuators, and this zone contains a region with the delamination ($x_{d1} \leqslant x \leqslant x_{d2}$), where the assumed transverse shear strain ε_{xz} has the form of Eq. (25), and two regions without delaminations ($x_1 \leqslant x < x_{d1}, x_{d2} \leqslant x \leqslant L$), where the assumed strain ε_{xz} has the form of Eq. (26). In order to represent the assumed strain ε_{xz} with a single expression, valid for both regions of the zone 2, with and without the delamination, we will use the extended Dirac's function, that can be defined by the formula:

$$\begin{aligned} D(x, x_{d1}, x_{d2}) &\equiv \frac{1}{\pi}\lim_{\beta\to 0}\left(\arctan\frac{x-x_{d1}}{\beta} - \arctan\frac{x-x_{d2}}{\beta}\right) \\ &= \frac{1}{2}\text{signum}(x-x_{d1}) + \frac{1}{2}\text{signum}(-x+x_{d2}) = \begin{cases} 1 & \text{for } x_{d1} < x < x_{d2}, \\ \frac{1}{2} & \text{for } x = x_{d1} \text{ or for } x = x_{d2}, \\ 0 & \text{for all other } x, \end{cases} \end{aligned} \tag{44}$$

where

$$\text{signum}(x) = \begin{cases} 0 & \text{for } x = 0 \\ \frac{x}{|x|} & \text{for all other } x. \end{cases} \tag{45}$$

Then, expressions (25) and (26) for ε_{xz} can be written as a single expression valid for the whole domain of the zone 2 of the beam:

$$\begin{aligned} 2\varepsilon_{xz} = {} & (1-D)\varphi_x\left(1-\frac{2}{h}z\right)\left(1+\frac{2}{h}z\right) + D\left[\varphi_x^{(1)}\left(1+\frac{2}{h}z\right)\left(1-\frac{z}{z_d}\right)(1-H) + \varphi_x^{(2)}\left(-1+\frac{2z}{h}\right)\right. \\ & \left. \times\left(1-\frac{z}{z_d}\right)H\right] \quad (x_1 \leqslant x \leqslant L). \end{aligned} \tag{46}$$

According to Eq. (20), the assumed displacement w in the region with the delamination is

$$w(x,y,z,t) = [1 - H(z,z_d)]W_0(x,y,t) + H(z,z_d)W_1(x,y,t) \quad (x_{d1} \leqslant x \leqslant x_{d2}).$$

According to Eq. (23), in the region of the zone 2 without the delamination, the assumed displacement w has the form

$$w(x,y,z,t) = W_0(x,y,t) \quad (x_1 \leqslant x < x_{d1},\ x_{d2} < x \leqslant L).$$

The last two equations can be combined into a single one that is valid for the whole domain of the zone 2:

$$w = (1-D)W_0 + D[(1-H)W_0 + HW_1] = W_0 + DH(W_1 - W_0) \quad (x_1 < x < L). \tag{47}$$

We now need to find an expression for the longitudinal displacement $u(x,z,t)$ such that it takes account of discontinuity of the displacement u at the surfaces of the delamination crack (at $z = z_d$) and satisfies the strain–displacement relation $2\varepsilon_{xz} = \partial u/\partial z + \partial w/\partial x$. If the delamination crack is above the axis of the beam, i.e. if $z_d > 0$, then the sought expression for $u(x,z,t)$ can be assumed to have the form

$$u(x,z,t) = U_0(x,t) + H(z,z_d)[U_1(x,t) - U_0(x,t)] + \int_0^z \left(2\varepsilon_{xz} - \frac{\partial w}{\partial x}\right) dz, \tag{48}$$

where $U_0(x,t) \equiv u|_{z=0}$, and $U_1(x,t)$ is a function that characterizes discontinuity of the displacement u at the surfaces of the delamination crack (i.e. at $z = z_d$). Substitution of Eq. (46) for the assumed ε_{xz} and Eq. (47) for the assumed w into Eq. (48) gives the result:

$$
\begin{aligned}
u = {} & (1-D)\left[U_0 - \frac{\partial W_0}{\partial x}z + \varphi_x\left(z - \frac{4}{3h^2}z^3\right)\right] + D\Bigg[(1-H)U_0 + HU_1 + \varphi_x^{(1)}(1-H) \\
& \times\left(-\frac{2}{3hz_d}z^3 + \frac{6z_d - 3h}{6hz_d}z^2 + z\right) + \varphi_x^{(2)}H\left(-\frac{2}{3hz_d}z^3 + \frac{6z_d + 3h}{6hz_d}z^2 - z\right) \\
& - \left((1-H)\frac{\partial W_0}{\partial x} + H\frac{\partial W_1}{\partial x}\right)z\Bigg] \quad (x_1 < x < L). \qquad (49)
\end{aligned}
$$

Substitution of Eq. (49) for u and Eq. (47) for w into the strain–displacement relation $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ gives an expression for the strain component ε_{xx}, in terms of the unknown functions

$$
\begin{aligned}
\varepsilon_{xx} = {} & (1-D)\left[\frac{\partial U_0}{\partial x} - \frac{\partial^2 W_0}{\partial x^2}z + \frac{\partial \varphi_x}{\partial x}\left(z - \frac{4}{3h^2}z^3\right)\right] + D\Bigg[(1-H)\frac{\partial U_0}{\partial x} + H\frac{\partial U_1}{\partial x} + \frac{\partial \varphi_x^{(1)}}{\partial x}(1-H) \\
& \times\left(-\frac{2}{3hz_d}z^3 + \frac{6z_d - 3h}{6hz_d}z^2 + z\right) + \frac{\partial \varphi_x^{(2)}}{\partial x}H\left(-\frac{2}{3hz_d}z^3 + \frac{6z_d + 3h}{6hz_d}z^2 - z\right) \\
& - \left((1-H)\frac{\partial^2 W_0}{\partial x^2} + H\frac{\partial^2 W_1}{\partial x^2}\right)z\Bigg]. \qquad (50)
\end{aligned}
$$

The virtual work principle for the zone 2 of the beam can be written similarly to that for the zone 1 of the beam (Eq. (33)):

$$
\begin{aligned}
& b\int_0^l \int_{-h/2}^{h/2}\left[\frac{1}{\overline{S}_{11}^{(b)}}\varepsilon_{xx}\delta\varepsilon_{xx} + \frac{1}{\overline{S}_{55}^{(b)}}2\varepsilon_{xz}2\delta\varepsilon_{xz}\right]\mathrm{d}z\,\mathrm{d}x + b\int_0^l \int_{-h/2}^{h/2}\rho^{(b)}\ddot{u}\delta u\,\mathrm{d}z\,\mathrm{d}x \\
& + b\int_0^l \int_{-h/2}^{h/2}\rho^{(b)}(g + \ddot{w})\delta w\,\mathrm{d}z\,\mathrm{d}x = 0. \qquad (51)
\end{aligned}
$$

In order to perform a finite element formulation for the zone 2 of the beam, we will represent the unknown functions $U_0(x,t)$, $U_1(x,t)$, $W_0(x,t)$, $W_1(x,t)$, $\varphi_x(x,t)$, $\varphi_x^{(1)}(x,t)$, $\varphi_x^{(2)}(x,t)$ by piecewise interpolation polynomials. In the expressions for the displacements and strains in terms of the unknown functions (Eqs. (46), (47), (49) and (50)) and, therefore, in the virtual work principle, the maximum orders of derivatives are

Function	Maximum order of derivatives
U_0	1
U_1	1
W_0	2
W_1	2
φ_x	1
$\varphi_x^{(1)}$	1
$\varphi_x^{(2)}$	1

In the problem under consideration, the interpolation polynomial will be chosen to be of the lowest allowable degree and the order of their derivatives' continuity. Therefore, we choose the first degree Lagrange polynomials to interpolate the unknown functions U_0, U_1, φ_x, $\varphi_x^{(1)}$, $\varphi_x^{(2)}$:

$$U_0 = \underset{(1\times 2)}{\lfloor M \rfloor}\underset{(2\times 1)}{\{\overline{U_0}\}}, \quad U_1 = \underset{(1\times 2)}{\lfloor M \rfloor}\underset{(2\times 1)}{\{\overline{U_1}\}}, \quad \varphi_x = \underset{(1\times 2)}{\lfloor M \rfloor}\underset{(2\times 1)}{\{\overline{\varphi_x}\}}, \quad \varphi_x^{(1)} = \underset{(1\times 2)}{\lfloor M \rfloor}\underset{(2\times 1)}{\left\{\overline{\varphi_x^{(1)}}\right\}}, \quad \varphi_x^{(2)} = \underset{(1\times 2)}{\lfloor M \rfloor}\underset{(2\times 1)}{\left\{\overline{\varphi_x^{(2)}}\right\}}, \tag{52}$$

where

$$\{\overline{U_0}\} \equiv \begin{Bmatrix} U_0(0) \\ U_0(l) \end{Bmatrix}, \quad \{\overline{U_1}\} \equiv \begin{Bmatrix} U_1(0) \\ U_1(l) \end{Bmatrix}, \quad \{\overline{\varphi_x}\} \equiv \begin{Bmatrix} \varphi_x(0) \\ \varphi_x(l) \end{Bmatrix}, \quad \left\{\overline{\varphi_x^{(1)}}\right\} \equiv \begin{Bmatrix} \varphi_x^{(1)}(0) \\ \varphi_x^{(1)}(l) \end{Bmatrix},$$
$$\left\{\overline{\varphi_x^{(2)}}\right\} \equiv \begin{Bmatrix} \varphi_x^{(2)}(0) \\ \varphi_x^{(2)}(l) \end{Bmatrix}, \tag{53}$$

and the matrix of shape functions $\lfloor M \rfloor$ is defined by Eq. (35). For interpolation of functions W_0 and W_1 we choose the Hermit polynomials of the third degree:

$$W_0 = \underset{(1\times 4)}{\lfloor N \rfloor}\underset{(4\times 1)}{\{\overline{W}_0\}}, \qquad W_1 = \underset{(1\times 4)}{\lfloor N \rfloor}\underset{(4\times 1)}{\{\overline{W_1}\}}, \tag{54}$$

where

$$\underset{(4\times 1)}{\{\overline{W}_0\}} = \begin{Bmatrix} W_0(0) \\ \frac{\partial W_0}{\partial x}(0) \\ W_0(l) \\ \frac{\partial W_0}{\partial x}(l) \end{Bmatrix}, \qquad \underset{(4\times 1)}{\{\overline{W_1}\}} = \begin{Bmatrix} W_1(0) \\ \frac{\partial W_1}{\partial x}(0) \\ W_1(l) \\ \frac{\partial W_1}{\partial x}(l) \end{Bmatrix}, \tag{55}$$

and the matrix of shape functions $\lfloor N \rfloor$ is defined by Eqs. (38) and (39).

The vector of nodal parameters of the finite element for the zone 2 of the beam is introduced as follows:

$$\begin{aligned} &\theta_1 = U_0(0),\ \theta_2 = U_1(0), \quad \theta_3 = W_0(0), \quad \theta_4 = \frac{\partial W_0}{\partial x}(0), \quad \theta_5 = W_1(0), \quad \theta_6 = \frac{\partial W_1}{\partial x}(0), \\ &\theta_7 = \varphi_x(0), \quad \theta_8 = \varphi_x^{(1)}(0), \quad \theta_9 = \varphi_x^{(2)}(0), \quad \theta_{10} = U_0(l), \quad \theta_{11} = U_1(l), \quad \theta_{12} = W_0(l), \\ &\theta_{13} = \frac{\partial W_0}{\partial x}(l), \quad \theta_{14} = W_1(l), \quad \theta_{15} = \frac{\partial W_1}{\partial x}(l), \quad \theta_{16} = \varphi_x(l), \quad \theta_{17} = \varphi_x^{(1)}(l), \quad \theta_{18} = \varphi_x^{(2)}(l). \end{aligned} \tag{56}$$

Then, equations of motion for a finite element of the zone 2 of the beam, derived from the virtual work principle (51) are

$$\underset{(18\times 18)}{[m]}\underset{(18\times 1)}{\{\ddot{\theta}\}} + \underset{(18\times 18)}{[k]}\underset{(18\times 1)}{\{\theta\}} = \underset{(18\times 1)}{\{q\}}, \tag{57}$$

where the matrices $[m]$, $[k]$ and the vector $\{f\}$ are presented in Appendix C.

4. Verification of results of the FE analysis

In order to verify an accuracy of finite element models based on the theory presented above, we considered several example problems for cantilever beams with piezoelectric actuators attached near the clamped edges (one actuator attached to each beam, Fig. 1).

In the first example problem we considered a wooden beam without a delamination, and in the FE model the transverse shear strain ε_{xz} was set equal to zero (by setting equal to zero the nodal parameters associated with the unknown functions φ_x, $\varphi_x^{(1)}$ and $\varphi_x^{(2)}$). The beam had the following characteristics: length of the beam $L = 430.4 \times 10^{-3}$ m, density of the wood: $\rho = 464.52$ kg/m^3, width of the beam $b = 3.81 \times 10^{-2}$ m, thickness of the beam $h = 1.9025 \times 10^{-2}$ m, Young's modulus of wood in the direction of the fibers: $E_1 = 1.0897 \times 10^{10}$ N/m^2, shear modulus $G_{13} = 0.43588 \times 10^{10}$ N/m^2. Elastic compliance coefficients in the laminate coordinate system, $\overline{S}_{11}$ and $\overline{S}_{55}$, are computed by the formulas (A.32) and (A.33). Edges of the wooden beam were cut along the visible direction of fibers, so $\theta = 0$, therefore

$$\overline{S}_{55} = \frac{1}{G_{13}} = 2.2942 \times 10^{-10}\ \mathrm{m^2/N},$$

$$\overline{S}_{11} = \frac{1}{E_1} = 0.91768 \times 10^{-10}\ \mathrm{m^2/N}.$$

In the finite element program, the length of the delamination crack was set equal to zero by setting equal the coordinates of the left and right tips of the delamination crack, i.e. $x_{\mathrm{d1}} = x_{\mathrm{d2}}$. An analytical formulation of this problem, for a beam without the delamination and with ε_{xz} set equal to zero, can be written in a form of the differential equation

$$E_1 I \frac{\partial^2 w}{\partial x^2} + \bar{m}\frac{\partial^2 w}{\partial t^2} = 0 \tag{58}$$

(where $\bar{m}$ is a mass of the beam per unit length, and I is a moment of inertia of a cross-section) with boundary conditions, which for the cantilever beam have the form

$$w(0) = 0, \qquad \frac{\partial w}{\partial x}(0) = 0. \tag{59}$$

The solution of the differential equation (58) with the boundary conditions (59) gives the following natural frequency $v = 1/T$

$$v_n = \frac{c_n^2}{2\pi}\sqrt{\frac{EI}{\bar{m}L^4}}, \tag{60}$$

where c_n is computed from the equation

$$\cos c_n \cosh c_n + 1 = 0. \tag{61}$$

The natural frequencies for the beam without delamination and without shear strain taken into account, computed analytically from Eq. (60) and with the use of the FE model with 20 elements (based on the FE formulation presented above) are shown in Table 1.

In the second example problem, the same wooden beam without a delamination was considered, but in the finite element model the transverse shear strain ε_{xz} was taken into account. The frequencies, computed with the use of the finite element model, were compared with the frequencies measured experimentally with the use of a laser vibrometer, PSV-300, a POLYTEC manufactured product. A size and weight of the

Table 1
Natural frequencies of a beam without debonding, transverse shear strain set equal to zero

v ($\mathrm{s^{-1}}$) from analytical solution	v ($\mathrm{s^{-1}}$) from FEA (20 elements)
80.354	81.7 (error 1.7%)
503.57	516 (error 2.5%)
1410	1518 (error 7.6%)
2763	2934 (error 6.1%)
4567	4490 (error 1.7%)
6823	5464 (error 20%)
9529	10,583 (error 11%)
12,688	13,727 (error 8.2%)
16,296	16,291 (error 0.03%)
20,357	20,000 (error 1.8%)

Table 2
Natural frequencies of a beam without debonding, in the FE model transverse shear strain taken into account

ν (s^{-1}) experimental	ν (s^{-1}) from FE solution (20 elements)
80	81 (error 1.3%)
1068	998 (error 6.6%)
3192	3041 (error 4.7%)
5242	5247 (error 0.1%)
7947	7565 (error 4.8%)
10,850	10,523 (error 3.01%)
12,710	12,885 (error 1.4%)
15,780	15,797 (error 0.1%)
15,910	16,054 (error 0.9%)
19,060	18,975 (error 0.4%)

Table 3
Natural frequencies of a beam with debonding, in the FE model transverse shear strain taken into account

ν (s^{-1}) experimental	ν (s^{-1}) from FE solution (20 elements)
85.9	86.4 (error 0.6%)
341	301 (error 11%)
3356	3287 (error 2%)
4810	4829 (error 0.4%)
6485	6365 (error 1.8%)
7793	7814 (error 0.3%)
8488	8591 (error 1.2%)
9119	9294 (error 1.9%)

piezoelectric actuator, attached to the upper surface of the beam near the clamp (Fig. 1), was very small as compared to the size and weight of the beam. Therefore, the influence of the presence of the actuator on the natural frequencies of the beam was negligibly small. The excitation voltage to the piezoelectric actuator contained frequencies in the range from 0 to 2×10^4 Hz. Results of this comparison are presented in Table 2.

In the third example, a wooden two-ply beam with debonding (delamination) between plies was considered. This two-ply beam was constructed by adhering two separate wooden plies. The delamination was permanently formed by removing the adhesion.

The material and geometric characteristics of this delaminated beam were the same as in the previous example problems. The delamination crack was parallel to the surfaces of the beam. The x-coordinates of the tips of the delamination crack were $x_{d1} = 149.2 \times 10^{-3}$ m and $x_{d2} = 353.15 \times 10^{-3}$ m, and z-coordinate of the delamination crack was $z_d = 3.34 \times 10^{-3}$ m. In the finite element model the transverse shear strain ε_{xz} was taken into account. A comparison of computed and experimentally measured frequencies is presented in Table 3. The higher computed frequencies are inaccurate.

5. Conclusion

A theory of beams with delaminations, presented in this paper, is developed for the use in model-supported damage identification. The frequencies computed with the use of the finite element model, based on the formulation presented in this paper, are in good agreement with experimental results. This shows the validity of the simplifying assumptions adopted in the present paper for constructing a 1-D theory of a beam with delamination. Therefore, a similar approach, with the use of the Heaviside function and extended Dirac function, can be attempted for constructing a 2-D plate theory.

Appendix A. Constitutive equations of an anisotropic piezoelectric

The stress, strain, electric field and electric displacement within a piezoelectric material can be fully described by a pair of electromechanical equations:

$$\varepsilon_{ij} = s_{ijkl}\sigma_{kl} + d_{kij}\mathscr{E}_k, \tag{A.1}$$

$$D_i = d_{ikl}\sigma_{kl} + \underbrace{\epsilon_0\epsilon_{ik}}_{\zeta_{ik}}\mathscr{E}_k, \tag{A.2}$$

where ε_{ij} is a strain tensor, σ_{ij} is a stress tensor, s_{ijkl} is a compliance tensor, d_{kij} is a tensor of piezoelectric constants, $\mathscr{E}_k$ is the electric field intensity applied to the piezoelectric element; D_i is the electric displacement, ϵ_{ik} is the dielectric permittivity tensor. In the subsequent text, the following notation will be introduced for simplicity

$$\zeta_{ik} \equiv \epsilon_0\epsilon_{ik}. \tag{A.3}$$

The second term in Eq. (A.1), $d_{kij}\mathscr{E}_k$, takes account of the converse piezoelectric effect, i.e. deformation under an applied external electric field. The first term in Eq. (A.2), $d_{ikl}\sigma_{kl}$, takes account of the direct piezoelectric effect, i.e. the fact that stresses (or strains) in piezoelectric dielectrics induce in them electric fields.

With the use of compact notations for stresses and strains, Eqs. (A.1) and (A.2) can be written as follows:

$$\varepsilon_p = S_{pq}\sigma_q + d_{kp}\mathscr{E}_k, \tag{A.4}$$

$$D_i = d_{iq}\sigma_q + \zeta_{ik}\mathscr{E}_k, \tag{A.5}$$

where $i = 1, 2, 3$; $k = 1, 2, 3$; $p = 1, \ldots, 6$; $q = 1, \ldots, 6$. In Eqs. (A.4) and (A.5), the quantities with indices do not transform as components of tensors. In matrix form, the constitutive equations (A.4) and (A.5) can be written as follows:

$$\underset{(6\times 1)}{\{\varepsilon\}} = \underset{(6\times 6)}{[S]}\underset{(6\times 1)}{\{\sigma\}} + \underset{(6\times 3)}{[d]}^{\mathrm{T}}\underset{(3\times 1)}{\{\mathscr{E}\}}, \tag{A.6}$$

$$\underset{(3\times 1)}{\{D\}} = \underset{(3\times 6)}{[d]}\underset{(6\times 1)}{\{\sigma\}} + \underset{(3\times 3)}{[\zeta]}\underset{(3\times 1)}{\{\mathscr{E}\}}. \tag{A.7}$$

Premultiplying Eq. (A.6) by $[C] \equiv [S]^{-1}$, we obtain

$$\underset{(6\times 1)}{\{\sigma\}} = \underset{(6\times 6)}{[C]}\underset{(6\times 1)}{\{\varepsilon\}} - \underset{(6\times 3)}{[e]}^{\mathrm{T}}\underset{(3\times 1)}{\{\mathscr{E}\}}, \tag{A.8}$$

where

$$\underset{(3\times 6)}{[e]} = \underset{(3\times 6)}{[d]}\underset{(6\times 6)}{[C]},$$

is called a piezoelectric stiffness matrix, and the corresponding transposed matrix is

$$\underset{(6\times 3)}{[e]}^{\mathrm{T}} = \underset{(6\times 6)}{[C]}^{\mathrm{T}}\underset{(6\times 3)}{[d]}^{\mathrm{T}} = \underset{(6\times 6)}{[C]}\underset{(6\times 3)}{[d]}^{\mathrm{T}}.$$

Eq. (A.8) can be substituted into Eq. (A.7) with the following result:

$$\underset{(3\times 1)}{\{D\}} = \underset{(3\times 6)}{[e]}\underset{(6\times 1)}{\{\varepsilon\}} + \underset{(3\times 3)}{[\xi]}\underset{(3\times 1)}{\{\mathscr{E}\}}, \tag{A.9}$$

where

$$\underset{(3\times 3)}{[\xi]} = \underset{(3\times 3)}{[\zeta]} - \underset{(3\times 6)}{[d]}\underset{(6\times 3)}{[e]}^{\mathrm{T}}.$$

In summary, the constitutive equations of a piezoelectric can be expressed either by the pair of matrix equations (A.6) and (A.7) or by the pair of matrix equations (A.8) and (A.9), with relations between the matrices being the following:

$$[C] \equiv [S]^{-1}, \qquad \underset{(3\times 6)}{[e]} = \underset{(3\times 6)}{[d]}\underset{(6\times 6)}{[C]}, \qquad \underset{(3\times 3)}{[\xi]} = \underset{(3\times 3)}{[\zeta]} - \underset{(3\times 6)}{[d]}\underset{(6\times 3)}{[e]}^{\mathrm{T}}. \tag{A.10}$$

We will consider a piezoelectric material with orthorhombic mm2 symmetry, such as PVDF or lead PZT. In the manufacturing process, the planes of elastic symmetry can be made the same as the planes of piezoelectric symmetry. In this case, the constitutive relations in the principle material coordinate system have the form (Mitchell and Reddy, 1994; Varadan et al., 1989; Nix and Ward, 1986; Dunn and Taya, 1993):

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 & 0 & 0 & e_{32} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 & 0 & 0 & e_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & e_{24} & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 & e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \xi_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 & 0 & \xi_{22} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 & 0 & 0 & \xi_{33} \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix}, \tag{A.11}$$

where the stiffness coefficients C_{ij} are written in terms of engineering constants as follows (Reddy, 1996):

$$C_{11} = \frac{(-E_2 + v_{23}^2 E_3)E_1^2}{\Delta}, \qquad C_{12} = \frac{-(v_{12}E_2 + v_{23}v_{13}E_3)E_1E_2}{\Delta},$$

$$C_{13} = \frac{-(v_{12}v_{23} + v_{13})E_2E_1E_3}{\Delta}, \qquad C_{22} = \frac{(-E_1 + v_{13}^2 E_3)E_2^2}{\Delta},$$

$$C_{23} = \frac{-(v_{23}E_1 + v_{13}v_{12}E_2)E_2E_3}{\Delta}, \qquad C_{33} = \frac{(-E_1 + v_{12}^2 E_2)E_2E_3}{\Delta},$$

$$C_{44} = G_{23}, \qquad C_{55} = G_{13}, \qquad C_{66} = G_{12}, \tag{A.12}$$

where

$$\Delta = -E_1E_2 + E_1v_{23}^2E_3 + v_{12}^2E_2^2 + 2v_{12}E_2v_{23}v_{13}E_3 + v_{13}^2E_2E_3. \tag{A.13}$$

If $\sigma_{zz} = 0$, then the equation for ε_{zz} in constitutive equations (A.11) can be discarded (because in case of $\sigma_{zz} = 0$, the term $(1/2)\sigma_{zz}\varepsilon_{zz}$ does not enter into the expression for the strain energy density, or expression $\sigma_{zz}\delta\varepsilon_{zz}$ does not enter into the expression for the virtual work), and, therefore, the third row and the third column in the 9×9 matrix of Eq. (A.11) can be deleted, leading to the following constitutive equations:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 & 0 & 0 & \varrho_{31} \\ Q_{12} & Q_{22} & 0 & 0 & 0 & 0 & 0 & \varrho_{32} \\ 0 & 0 & Q_{44} & 0 & 0 & 0 & \varrho_{24} & 0 \\ 0 & 0 & 0 & Q_{55} & 0 & \varrho_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varrho_{15} & 0 & \varsigma_{11} & 0 & 0 \\ 0 & 0 & \varrho_{24} & 0 & 0 & 0 & \varsigma_{22} & 0 \\ \varrho_{31} & \varrho_{32} & 0 & 0 & 0 & 0 & 0 & \varsigma_{33} \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix}, \tag{A.14}$$

where

$$Q_{11} = \frac{E_1^2}{E_1 - \nu_{12}^2 E_2}, \quad Q_{12} = \frac{\nu_{12} E_1 E_2}{E_1 - \nu_{12}^2 E_2}, \quad Q_{22} = \frac{E_1 E_2}{E_1 - \nu_{12}^2 E_2}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}, \quad Q_{66} = G_{12},$$

$$\varrho_{31} = \frac{E_1 e_{33} \nu_{12} \nu_{23} + E_1 e_{33} \nu_{13} - e_{31} E_1 + e_{31} \nu_{12}^2 E_2}{\nu_{12}^2 E_2 - E_1}, \quad \varrho_{32} = \frac{e_{33} \nu_{23} E_1 + e_{33} \nu_{13} \nu_{12} E_2 - e_{32} E_1 + e_{32} \nu_{12}^2 E_2}{\nu_{12}^2 E_2 - E_1},$$

$$\varrho_{24} = e_{24}, \quad \varrho_{15} = e_{15}, \qquad \varsigma_{11} = \xi_{11}, \quad \varsigma_{22} = \xi_{22},$$

$$\varsigma_{33} = \frac{E_2 \nu_{12}^2}{E_1 - \nu_{12}^2 E_2} \xi_{33} + \frac{E_2 \nu_{12}^2}{E_3 (E_1 - \nu_{12}^2 E_2)} e_{33}^2 + \frac{2 \nu_{13} \nu_{23} \nu_{12}}{E_1 - \nu_{12}^2 E_2} e_{33}^2 + \frac{\nu_{13}^2}{E_1 - \nu_{12}^2 E_2} e_{33}^2 + \frac{E_1}{E_1 - \nu_{12}^2 E_2} \xi_{33}$$
$$+ \frac{\nu_{23}^2 E_1}{E_2 (E_1 - \nu_{12}^2 E_2)} e_{33}^2 + \frac{E_1}{E_3 (\nu_{12}^2 E_2 - E_1)} e_{33}^2. \tag{A.15}$$

In the problem coordinate system, rotated by an angle θ counterclockwise with respect to the principle material coordinate system, the constitutive equations have the form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \\ D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & 0 & 0 & 0 & \bar{\varrho}_{31} \\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & 0 & 0 & 0 & \bar{\varrho}_{32} \\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 & \bar{\varrho}_{14} & \bar{\varrho}_{24} & 0 \\ 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 & \bar{\varrho}_{15} & \bar{\varrho}_{25} & 0 \\ 0 & 0 & 0 & 0 & \overline{Q}_{66} & 0 & 0 & \bar{\varrho}_{36} \\ 0 & 0 & \bar{\varrho}_{14} & \bar{\varrho}_{15} & 0 & \bar{\varsigma}_{11} & \bar{\varsigma}_{12} & 0 \\ 0 & 0 & \bar{\varrho}_{24} & \bar{\varrho}_{25} & 0 & \bar{\varsigma}_{12} & \bar{\varsigma}_{22} & 0 \\ \bar{\varrho}_{31} & \bar{\varrho}_{32} & 0 & 0 & \bar{\varrho}_{36} & 0 & 0 & \bar{\varsigma}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \\ \mathscr{E}_x \\ \mathscr{E}_y \\ \mathscr{E}_z \end{Bmatrix}, \tag{A.16}$$

where, with the use of notations $c = \cos\theta$, $s = \sin\theta$, the transformed reduced stiffness coefficients $\overline{Q}_{ij}$, piezoelectric stiffness coefficients in case of $\sigma_{zz} = 0$, $\bar{\varrho}_{ij}$, and permittivity coefficients in case of $\sigma_{zz} = 0$, $\bar{\varsigma}_{ij}$, are written in terms of the corresponding quantities in the principle material coordinate system as follows (Reddy, 1999):

$$\begin{aligned}
&\overline{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}s^4,\\
&\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(s^4 + c^4),\\
&\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})sc^3 + (Q_{12} - Q_{22} + 2Q_{66})s^3c,\\
&\overline{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}c^4,\\
&\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})s^3c + (Q_{12} - Q_{22} + 2Q_{66})sc^3,\\
&\overline{Q}_{44} = Q_{44}c^2 + Q_{55}s^2, \quad \overline{Q}_{45} = (Q_{55} - Q_{44})cs,\\
&\overline{Q}_{55} = Q_{55}c^2 + Q_{44}s^2, \quad \overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4),\\
&\bar{\varrho}_{31} = \varrho_{31}c^2 + \varrho_{32}s^2, \quad \bar{\varrho}_{32} = \varrho_{31}s^2 + \varrho_{32}c^2, \quad \bar{\varrho}_{36} = (\varrho_{31} - \varrho_{32})cs,\\
&\bar{\varrho}_{14} = (\varrho_{15} - \varrho_{24})cs, \quad \bar{\varrho}_{24} = \varrho_{24}c^2 + \varrho_{15}s^2,\\
&\bar{\varrho}_{15} = \varrho_{15}c^2 + \varrho_{24}s^2, \quad \bar{\varrho}_{25} = (\varrho_{15} - \varrho_{24})cs,\\
&\bar{\varsigma}_{11} = \varsigma_{11}c + \varsigma_{22}s, \quad \bar{\varsigma}_{22} = \varsigma_{11}s + \varsigma_{22}c,\\
&\bar{\varsigma}_{33} = \varsigma_{33}, \quad \bar{\varsigma}_{12} = (\varsigma_{11} - \varsigma_{22})cs.
\end{aligned} \tag{A.17}$$

A.1. Constitutive equations for a beam

For a beam that bends in the x–z plane, we can assume

$$\sigma_{yy} = \sigma_{yz} = \sigma_{xy} = 0 \tag{A.18}$$

in addition to the assumption $\sigma_{zz} = 0$ adopted for the plate. In this case it is more convenient to use constitutive relations in the form of Eqs. (A.6) and (A.7). For an orthotropic material, these constitutive equations, in the principle material coordinate system, have the form

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} + \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \\ d_{14} & d_{24} & d_{34} \\ d_{15} & d_{25} & d_{35} \\ d_{16} & d_{26} & d_{36} \end{bmatrix} \begin{Bmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \end{Bmatrix}, \tag{A.19}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} + \begin{bmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} \end{bmatrix} \begin{Bmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \end{Bmatrix}, \tag{A.20}$$

where the compliance coefficients S_{ij} are expressed in terms of engineering constants by the formulas

$$\begin{aligned}
&S_{11} = \frac{1}{E_1}, \quad S_{12} = -\frac{v_{12}}{E_1}, \quad S_{13} = -\frac{v_{13}}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{23} = -\frac{v_{23}}{E_2}, \quad S_{33} = \frac{1}{E_3},\\
&S_{44} = \frac{1}{G_{23}}, \quad S_{55} = \frac{1}{G_{13}}, \quad S_{66} = \frac{1}{G_{12}}.
\end{aligned} \tag{A.21}$$

In the laminate coordinate system, rotated clockwise by angle θ with respect to the principle material coordinate system, the constitutive equations have the form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & 0 & 0 & 0 \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & 0 & 0 & 0 \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{S}_{44} & \overline{S}_{45} & 0 \\ 0 & 0 & 0 & \overline{S}_{45} & \overline{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} + \begin{bmatrix} \bar{d}_{11} & \bar{d}_{21} & \bar{d}_{31} \\ \bar{d}_{12} & \bar{d}_{22} & \bar{d}_{32} \\ \bar{d}_{13} & \bar{d}_{23} & \bar{d}_{33} \\ \bar{d}_{14} & \bar{d}_{24} & \bar{d}_{34} \\ \bar{d}_{15} & \bar{d}_{25} & \bar{d}_{35} \\ \bar{d}_{16} & \bar{d}_{26} & \bar{d}_{36} \end{bmatrix} \begin{Bmatrix} \mathscr{E}_x \\ \mathscr{E}_y \\ \mathscr{E}_z \end{Bmatrix}, \tag{A.22}$$

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} \bar{d}_{11} & \bar{d}_{12} & \bar{d}_{13} & \bar{d}_{14} & \bar{d}_{15} & \bar{d}_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} + \begin{bmatrix} \overline{\zeta}_{11} & \overline{\zeta}_{12} & \overline{\zeta}_{13} \\ \overline{\zeta}_{21} & \overline{\zeta}_{22} & \overline{\zeta}_{23} \\ \overline{\zeta}_{31} & \overline{\zeta}_{32} & \overline{\zeta}_{33} \end{bmatrix} \begin{Bmatrix} \mathscr{E}_x \\ \mathscr{E}_y \\ \mathscr{E}_z \end{Bmatrix}, \tag{A.23}$$

where the matrices in the problem coordinate system are expressed in terms of matrices in the material coordinate systems as follows (with the use of notation $c = \cos\theta$, $s = \sin\theta$):

$$\begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & 0 & 0 & 0 \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & 0 & 0 & 0 \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{S}_{44} & \overline{S}_{45} & 0 \\ 0 & 0 & 0 & \overline{S}_{45} & \overline{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{S}_{66} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -sc \\ s^2 & c^2 & 0 & 0 & 0 & sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2sc & -2sc & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}$$
$$\times \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2sc \\ s^2 & c^2 & 0 & 0 & 0 & -2sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -sc & sc & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \tag{A.24}$$

$$\begin{bmatrix} \overline{\zeta}_{11} & \overline{\zeta}_{12} & \overline{\zeta}_{13} \\ \overline{\zeta}_{21} & \overline{\zeta}_{22} & \overline{\zeta}_{23} \\ \overline{\zeta}_{31} & \overline{\zeta}_{32} & \overline{\zeta}_{33} \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{A.25}$$

$$\begin{bmatrix} \bar{d}_{11} & \bar{d}_{21} & \bar{d}_{31} \\ \bar{d}_{12} & \bar{d}_{22} & \bar{d}_{32} \\ \bar{d}_{13} & \bar{d}_{23} & \bar{d}_{33} \\ \bar{d}_{14} & \bar{d}_{24} & \bar{d}_{34} \\ \bar{d}_{15} & \bar{d}_{25} & \bar{d}_{35} \\ \bar{d}_{16} & \bar{d}_{26} & \bar{d}_{36} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -sc \\ s^2 & c^2 & 0 & 0 & 0 & sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2sc & -2sc & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \times \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \\ d_{14} & d_{24} & d_{34} \\ d_{15} & d_{25} & d_{35} \\ d_{16} & d_{26} & d_{36} \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A.26}$$

If $\sigma_{xz} = \sigma_{yz} = \sigma_{yy} = 0$ and $\mathscr{E}_x = \mathscr{E}_y = 0$ (these are the assumptions for a beam), then Eqs. (A.22) and (A.23) take the form

$$\begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & 0 \\ 0 & \overline{S}_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} + \begin{bmatrix} \bar{d}_{31} \\ \bar{d}_{35} \end{bmatrix} \{ \mathscr{E}_z \}, \tag{A.27}$$

$$\{ D_z \} = \begin{bmatrix} \bar{d}_{31} & \bar{d}_{35} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} + [\overline{\zeta}_{33}] \{ \mathscr{E}_z \}. \tag{A.28}$$

In the last two matrix equations, the rows corresponding to ε_{xz}, ε_{yz}, ε_{yy}, D_x, D_y are discarded because these quantities do not enter the virtual work principle if $\sigma_{xz} = \sigma_{yz} = \sigma_{yy} = 0$ and $\mathscr{E}_x = \mathscr{E}_y = 0$. From Eqs. (A.27) and (A.28) we can obtain the following relationships

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \left(\begin{bmatrix} \frac{1}{\overline{S}_{11}} & 0 \\ 0 & \frac{1}{\overline{S}_{55}} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \end{Bmatrix} - \begin{Bmatrix} \frac{\bar{d}_{31}}{\overline{S}_{11}} \\ \frac{\bar{d}_{35}}{\overline{S}_{55}} \end{Bmatrix} \mathscr{E}_z \right), \tag{A.29}$$

$$\{ D_z \} = \begin{bmatrix} \frac{d_{31}}{\overline{S}_{11}} & \frac{d_{35}}{\overline{S}_{55}} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \end{Bmatrix} + \left(\overline{\zeta}_{33} - \frac{d_{31}^2}{\overline{S}_{11}} - \frac{d_{35}^2}{\overline{S}_{55}} \right) \mathscr{E}_z. \tag{A.30}$$

Eqs. (A.29) and (A.30) can be written as a one matrix equation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \frac{1}{\overline{S}_{11}} & 0 & -\frac{\bar{d}_{31}}{\overline{S}_{11}} \\ 0 & \frac{1}{\overline{S}_{55}} & -\frac{\bar{d}_{35}}{\overline{S}_{55}} \\ \frac{\bar{d}_{31}}{\overline{S}_{11}} & \frac{\bar{d}_{35}}{\overline{S}_{55}} & \left(\overline{\zeta}_{33} - \frac{\bar{d}_{31}^2}{\overline{S}_{11}} - \frac{\bar{d}_{35}^2}{\overline{S}_{55}} \right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \mathscr{E}_z \end{Bmatrix}, \tag{A.31a}$$

or, in view of the relationship $\mathscr{E}_z = -\partial\varphi/\partial z$,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \frac{1}{\overline{S}_{11}} & 0 & \frac{\bar{d}_{31}}{\overline{S}_{11}} \\ 0 & \frac{1}{\overline{S}_{55}} & \frac{\bar{d}_{35}}{\overline{S}_{55}} \\ \frac{\bar{d}_{31}}{\overline{S}_{11}} & \frac{\bar{d}_{35}}{\overline{S}_{55}} & \left(-\overline{\zeta}_{33} + \frac{\bar{d}_{31}^2}{\overline{S}_{11}} + \frac{\bar{d}_{35}^2}{\overline{S}_{55}} \right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \frac{\partial\varphi}{\partial z} \end{Bmatrix}. \tag{A.31b}$$

According to Eqs. (A.21) and (A.24), the compliance coefficients in the problem coordinate system, $\overline{S}_{11}$ and $\overline{S}_{55}$, that enter into Eqs. (A.31a) and (A.31b), are expressed in terms of the engineering constants by the formulas

$$\overline{S}_{55} = \frac{1}{G_{23}} s^2 + \frac{1}{G_{13}} c^2, \tag{A.32}$$

$$\overline{S}_{11} = \frac{1}{E_1} c^4 + \frac{1}{E_2} s^4 + \left(\frac{1}{G_{12}} - 2\frac{v_{12}}{E_1} \right) s^2 c^2. \tag{A.33}$$

The constants $\bar{d}_{31}$ and $\bar{d}_{35}$ that characterize the piezoelectric properties in the problem coordinate system are expressed in terms of the constants d_{ij} in the material coordinate system by the formulas (derived from matrix equation (A.26))

$$\bar{d}_{31} = d_{31}c^2 + d_{32}s^2 - d_{36}sc, \qquad \overline{d}_{35} = -d_{34}s + d_{35}c, \tag{A.34}$$

and, according to the transformation equation (A.25),

$$\overline{\zeta}_{33} = \zeta_{33}. \tag{A.35}$$

For materials with orthorhombic mm2 symmetry, such as PVDF or lead-zirconate (PZT), a piezoelectric constant $\bar{d}_{35}$ of Eqs. (A.31a) and (A.31b) is equal to zero. Therefore, Eq. (A.31b) takes the form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \frac{1}{\overline{S}_{11}} & 0 & \frac{\bar{d}_{31}}{\overline{S}_{11}} \\ 0 & \frac{1}{\overline{S}_{55}} & 0 \\ \frac{\bar{d}_{31}}{\overline{S}_{11}} & 0 & \left(-\overline{\zeta}_{33} + \frac{\bar{d}_{31}^2}{\overline{S}_{11}} \right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \frac{\partial \varphi}{\partial z} \end{Bmatrix}. \tag{A.36}$$

Appendix B. Matrices of the finite element for the zone 1 of the beam (with the piezoelectric actuator attached to the upper surface and without delaminations)

Expressions for the stiffness matrix, mass matrix and load vector of the finite element will be written with the use of an auxiliary matrix $[\widetilde{G}]$, defined as follows

$$\underset{(3\times8)}{[\widetilde{G}]} = \underset{(3\times8)}{[\widetilde{Q}]}\,\underset{(8\times8)}{[\widetilde{R}]}, \tag{B.1}$$

where

$$\underset{(8\times8)}{[\widetilde{R}]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B.2}$$

$$\underset{(3\times8)}{[\widetilde{Q}]} = \begin{bmatrix} \underset{(1\times2)}{\lfloor M \rfloor} & \underset{(1\times4)}{\lfloor 0 \rfloor} & \underset{(1\times2)}{\lfloor 0 \rfloor} \\ \underset{(1\times2)}{\lfloor 0 \rfloor} & \underset{(1\times4)}{\lfloor N \rfloor} & \underset{(1\times2)}{\lfloor 0 \rfloor} \\ \underset{(1\times2)}{\lfloor 0 \rfloor} & \underset{(1\times4)}{\lfloor 0 \rfloor} & \underset{(1\times2)}{\lfloor M \rfloor} \end{bmatrix}, \tag{B.3}$$

$$\underset{(1\times2)}{\lfloor M \rfloor} = \left\lfloor 1 - \frac{\bar{x}}{l} \quad \frac{\bar{x}}{l} \right\rfloor, \tag{B.4}$$

$$\underset{(1\times4)}{\lfloor N \rfloor} \equiv \left\lfloor 1 - \frac{3\bar{x}^2}{l^2} + \frac{2\bar{x}^3}{l^3} \quad \bar{x} - \frac{2\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2} \quad \frac{3\bar{x}^2}{l^2} - \frac{2\bar{x}^3}{l^3} \quad -\frac{\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2} \right\rfloor. \tag{B.5}$$

Then the element mass matrix is

$$\underset{(8\times 8)}{[\tilde{m}]} = b\int_0^l \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 1 \\ 0 & 0 & \frac{2t}{(h+2t)h} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\right)^{\mathrm{T}}\left(\int_{-h/2}^{h/2}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}\rho^{(\mathrm{b})}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}^{\mathrm{T}}\mathrm{d}z\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 1 \\ 0 & 0 & \frac{2t}{(h+2t)h} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}$$
$$+ b\int_0^l \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 1 \\ 0 & 0 & \frac{2t}{(h+2t)h} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\right)^{\mathrm{T}}\left(\int_{h/2}^{t}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}\rho^{(\mathrm{p})}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}^{\mathrm{T}}\mathrm{d}z\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 1 \\ 0 & 0 & \frac{2t}{(h+2t)h} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}$$
$$+ b\left(\int_{-h/2}^{h/2}\rho^{(\mathrm{b})}\,\mathrm{d}z\right)\int_0^l \underset{(8\times 3)}{[\widetilde{G}]^{\mathrm{T}}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x} + b\left(\int_{h/2}^{t}\rho^{(\mathrm{p})}\,\mathrm{d}z\right)\int_0^l \underset{(8\times 3)}{[\widetilde{G}]^{\mathrm{T}}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}, \tag{B.6}$$

the element stiffness matrix is

$$\underset{(8\times 8)}{[\tilde{k}]} = b\int_0^l \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{2t}{(h+2t)h}\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h}\frac{\partial}{\partial x} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\right)^{\mathrm{T}}\left(\int_{-h/2}^{h/2}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}\frac{1}{\overline{S}_{11}^{(\mathrm{b})}}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}^{\mathrm{T}}\mathrm{d}z\right)\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{2t}{(h+2t)h}\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h}\frac{\partial}{\partial x} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}$$
$$+ b\int_0^l \underset{(8\times 3)}{[\widetilde{G}]^{\mathrm{T}}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{2}{h}-\frac{2}{h+2t} & -\frac{4}{(h+2t)h} \end{bmatrix}\int_{-h/2}^{h/2}\left(\begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix}\frac{1}{\overline{S}_{55}^{(\mathrm{b})}}\lfloor 1 \quad z \quad z^2 \rfloor\right)\mathrm{d}z\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{2}{h}-\frac{2}{h+2t} \\ 0 & 0 & -\frac{4}{(h+2t)h} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}$$
$$+ b\int_0^l \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{2t}{(h+2t)h}\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h}\frac{\partial}{\partial x} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\right)^{\mathrm{T}}\int_{h/2}^{t}\left(\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}\frac{1}{\overline{S}_{11}^{(\mathrm{p})}}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}^{\mathrm{T}}\right)\mathrm{d}z\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{2t}{(h+2t)h}\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h}\frac{\partial}{\partial x} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}$$
$$+ b\int_0^l \underset{(8\times 3)}{[\widetilde{G}]^{\mathrm{T}}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{2}{h}-\frac{2}{h+2t} & -\frac{4}{(h+2t)h} \end{bmatrix}\left(\int_{h/2}^{t}\begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix}\frac{1}{\overline{S}_{55}^{(\mathrm{p})}}\lfloor 1 \quad z \quad z^2 \rfloor\,\mathrm{d}z\right)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{2}{h}-\frac{2}{h+2t} \\ 0 & 0 & -\frac{4}{(h+2t)h} \end{bmatrix}\underset{(3\times 8)}{[\widetilde{G}]}\,\mathrm{d}\bar{x}, \tag{B.7}$$

the vector part of the element force vector $\underset{(8\times1)}{\{\widetilde{q}\}}V(\tau)$ is

$$\underset{(8\times1)}{\{\widetilde{q}\}} = -\frac{b\overline{d_{31}}}{t}\int_0^l\left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{2t}{(h+2t)h}\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{4}{3}\frac{1}{(h+2t)h}\frac{\partial}{\partial x} \end{bmatrix}\underset{(3\times8)}{[\widetilde{G}]}\right)^{\mathrm{T}}\mathrm{d}\bar{x}\left(\int_{h/2}^{t}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix}\left(-\frac{1}{\overline{S}_{11}^{(\mathrm{p})}}\right)\mathrm{d}z\right)$$
$$- bg\left(\int_{-h/2}^{h/2}\rho^{(\mathrm{b})}\,\mathrm{d}z\right)\int_0^l\underset{(8\times3)}{[\widetilde{G}]^{\mathrm{T}}}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}\mathrm{d}\bar{x} - bg\left(\int_{h/2}^{t}\rho^{(\mathrm{p})}\,\mathrm{d}z\right)\int_0^l\underset{(8\times3)}{[\widetilde{G}]^{\mathrm{T}}}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}\mathrm{d}\bar{x}. \tag{B.8}$$

Appendix C. Matrices of the finite element for the zone 2 of the beam (without piezoelectric actuators and with delaminations)

Expressions for element matrices $[m]$, $[k]$, $\{r_{nl}\}$ and $\{f\}$ are written in this appendix with the use of the following auxiliary matrices

$$\underset{(7\times18)}{[Q]} = \begin{bmatrix} \underset{(1\times2)}{\lfloor M\rfloor} & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor \\ \lfloor 0\rfloor & \underset{(1\times2)}{\lfloor M\rfloor} & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor \\ \lfloor 0\rfloor & \lfloor 0\rfloor & \underset{(1\times4)}{\lfloor N\rfloor} & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor \\ \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \underset{(1\times4)}{\lfloor N\rfloor} & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor \\ \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \underset{(1\times2)}{\lfloor M\rfloor} & \lfloor 0\rfloor & \lfloor 0\rfloor \\ \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \underset{(1\times2)}{\lfloor M\rfloor} & \lfloor 0\rfloor \\ \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \lfloor 0\rfloor & \underset{(1\times2)}{\lfloor M\rfloor} \end{bmatrix}, \tag{C.1}$$

where

$$\underset{(1\times2)}{\lfloor M\rfloor} = \left\lfloor 1-\frac{\bar{x}}{l} \quad \frac{\bar{x}}{l}\right\rfloor,$$

$$\underset{(1\times4)}{\lfloor N\rfloor} \equiv \left\lfloor 1-\frac{3\bar{x}^2}{l^2}+\frac{2\bar{x}^3}{l^3} \quad \bar{x}-\frac{2\bar{x}^2}{l}+\frac{\bar{x}^3}{l^2} \quad \frac{3\bar{x}^2}{l^2}-\frac{2\bar{x}^3}{l^3} \quad -\frac{\bar{x}^2}{l}+\frac{\bar{x}^3}{l^2}\right\rfloor.$$

$$\underset{(18\times18)}{[R]} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}, \tag{C.2}$$

$$\underset{(7\times 18)}{[G]} = \underset{(7\times 18)}{[Q]}\underset{(18\times 18)}{[R]}, \tag{C.3}$$

$$\underset{(4\times 7)}{[\partial_u]} = \begin{bmatrix} 1-DH & DH & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (D-H)\frac{\partial}{\partial x} & H\frac{\partial}{\partial x} & (1-D) & (1-H) & -H \\ 0 & 0 & 0 & 0 & 0 & (1-H)\frac{6z_d-3h}{6hz_d} & H\frac{6z_d+3h}{6hz_d} \\ 0 & 0 & 0 & 0 & -\frac{4}{3h^2}(1-D) & -\frac{2}{3hz_d}(1-H) & -\frac{2}{3hz_d}H \end{bmatrix}, \tag{C.4}$$

$$\underset{(1\times 7)}{[\partial_w]} = \begin{Bmatrix} 0 \\ 0 \\ 1-DH \\ DH \\ 0 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}}, \tag{C.5}$$

$$\underset{(4\times 7)}{[\partial_{\varepsilon_{xx}}]} = \begin{bmatrix} (1-DH)\frac{\partial}{\partial x} & 0 & 0 & 0 \\ DH\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & (D-H)\frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & H\frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & (1-D)\frac{\partial}{\partial x} & 0 & -\frac{4}{3h^2}(1-D)\frac{\partial}{\partial x} \\ 0 & (1-H)\frac{\partial}{\partial x} & (1-H)\frac{6z_d-3h}{6hz_d}\frac{\partial}{\partial x} & -\frac{2}{3hz_d}(1-H)\frac{\partial}{\partial x} \\ 0 & -H\frac{\partial}{\partial x} & H\frac{6z_d+3h}{6hz_d}\frac{\partial}{\partial x} & -\frac{2}{3hz_d}H\frac{\partial}{\partial x} \end{bmatrix}^{\mathrm{T}}, \tag{C.6}$$

$$\underset{(3\times 7)}{[\partial_{\varepsilon_{xz}}]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (D-H+1-DH)\frac{\partial}{\partial x} & 0 & 0 \\ H(1+D)\frac{\partial}{\partial x} & 0 & 0 \\ 1-D & 0 & -\frac{4}{3h^2}(1-D) \\ 1-H & \frac{1}{6}(1-H)\frac{6z_d-3h}{hz_d} & -\frac{2}{3hz_d}(1-H) \\ -H & \frac{1}{6}H\frac{6z_d+3h}{hz_d} & -\frac{2}{3hz_d}H \end{bmatrix}^{\mathrm{T}}. \tag{C.7}$$

$$\underset{(4\times 18)}{[B_u]} = \underset{(4\times 7)}{[\partial_u]}\underset{(7\times 18)}{[G]}, \tag{C.8}$$

$$\underset{(1\times 18)}{[B_w]} = \underset{(1\times 7)}{[\partial_w]}\underset{(7\times 18)}{[G]}, \tag{C.9}$$

$$\underset{(4\times 18)}{[B_{\varepsilon_{xx}}]} = \underset{(4\times 7)}{[\partial_{\varepsilon_{xx}}]}\underset{(7\times 18)}{[G]}, \tag{C.10}$$

$$\underset{(3\times 18)}{[B_{\varepsilon_{xz}}]} = \underset{(3\times 7)}{[\partial_{\varepsilon_{xz}}]}\underset{(7\times 18)}{[G]}, \tag{C.11}$$

$$\underset{(4\times 4)}{[A]} = \lfloor 1 \quad z \quad z^2 \quad z^3 \rfloor^{\mathrm{T}} \frac{1}{\overline{S}_{11}} \lfloor 1 \quad z \quad z^2 \quad z^3 \rfloor, \tag{C.12}$$

$$\underset{(4\times1)}{\{\widetilde{A}\}} = \lfloor 1 \quad z \quad z^2 \quad z^3 \rfloor^{\mathrm{T}} \frac{1}{\overline{S}_{11}}, \tag{C.13}$$

$$[\check{A}] = \lfloor 1 \quad 2z \quad 3z^2 \rfloor^{\mathrm{T}} \frac{1}{\overline{S}_{55}} \lfloor 1 \quad 2z \quad 3z^2 \rfloor, \tag{C.14}$$

$$[\overline{A}] = \left[1 \quad z \quad z^2 \quad z^3\right]^{\mathrm{T}} \rho \left[1 \quad z \quad z^2 \quad z^3\right]. \tag{C.15}$$

Then, the element stiffness matrix is

$$\begin{aligned}\underset{(18\times18)}{[k]} &= b\int_0^l \int_{-h/2}^{h/2} \underset{(18\times4)}{[B_{\varepsilon_{xx}}]^{\mathrm{T}}} \underset{(4\times4)}{[A]} \underset{(4\times18)}{[B_{\varepsilon_{xx}}]} \,\mathrm{d}z\,\mathrm{d}\bar{x} + b\int_0^l \int_{-h/2}^{h/2} \underset{(18\times3)}{[B_{\varepsilon_{xz}}]^{\mathrm{T}}} \underset{(3\times3)}{[\check{A}]} \underset{(3\times18)}{[B_{\varepsilon_{xz}}]} \,\mathrm{d}z\,\mathrm{d}\bar{x} \\ &= b\int_0^l \left(\sum_{k=1}^{N} \int_{\zeta_k}^{\zeta_{k+1}} \underset{(18\times4)}{[B_{\varepsilon_{xx}}]^{\mathrm{T}}} \underset{(4\times4)}{[A]} \underset{(4\times18)}{[B_{\varepsilon_{xx}}]} \,\mathrm{d}z\right)\mathrm{d}\bar{x} + b\int_0^l \left(\sum_{k=1}^{N} \int_{\zeta_k}^{\zeta_{k+1}} \underset{(18\times3)}{[B_{\varepsilon_{xz}}]^{\mathrm{T}}} \underset{(3\times3)}{[\check{A}]} \underset{(3\times18)}{[B_{\varepsilon_{xz}}]} \,\mathrm{d}z\right)\mathrm{d}\bar{x},\end{aligned} \tag{C.16}$$

where ζ_k are coordinates of interfaces between plies of the composite beam, N is a number of plies in the composite beam;

the element mass matrix is

$$\begin{aligned}\underset{(18\times18)}{[m]} &= b\int_0^l \int_{-h/2}^{h/2} \underset{(18\times4)}{[B_u]^{\mathrm{T}}} \underset{(4\times4)}{[\overline{A}]} \underset{(4\times18)}{[B_u]} \,\mathrm{d}z\,\mathrm{d}\bar{x} + b\int_0^l \int_{-h/2}^{h/2} \underset{(18\times1)}{[B_w]^{\mathrm{T}}} \rho \underset{(1\times18)}{[B_w]} \,\mathrm{d}z\,\mathrm{d}\bar{x} \\ &= b\int_0^l \sum_{k=1}^{N} \left(\int_{\zeta_k}^{\zeta_{k+1}} \underset{(18\times4)}{[B_u]^{\mathrm{T}}} \underset{(4\times4)}{[\overline{A}]} \underset{(4\times18)}{[B_u]} \,\mathrm{d}z\right)\mathrm{d}\bar{x} + b\int_0^l \left(\sum_{k=1}^{N} \int_{\zeta_k}^{\zeta_{k+1}} \underset{(18\times1)}{[B_w]} \rho \underset{(1\times18)}{[B_w]} \,\mathrm{d}z\right)\mathrm{d}\bar{x}.\end{aligned} \tag{C.17}$$

the element force vector is

$$\underset{(18\times1)}{\{f\}} = -bg\int_0^l \int_{-h/2}^{h/2} \underset{(18\times1)}{[B_w]^{\mathrm{T}}} \rho \,\mathrm{d}z\,\mathrm{d}\bar{x} = bg\int_0^l \left(\sum_{k=1}^{N} \int_{\zeta_k}^{\zeta_{k+1}} \underset{(18\times1)}{[B_w]^{\mathrm{T}}} \rho \,\mathrm{d}z\right)\mathrm{d}\bar{x}. \tag{C.18}$$

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